

Midterm #2 – Practice

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches or any other posted material). Any typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Problem 1. Before working on the problems below, make sure that you have completed all homework.

Problem 2. Solve the initial value problem $\mathbf{y}' = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix} \mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Solution.

- $A = \begin{bmatrix} 4 & -8 \\ -1 & 6 \end{bmatrix}$ has characteristic polynomial $(4 - \lambda)(6 - \lambda) - 8 = \lambda^2 - 10\lambda + 16 = (\lambda - 2)(\lambda - 8)$.

Hence, the eigenvalues of A are 2, 8.

The 8-eigenspace $\text{null}\left(\begin{bmatrix} -4 & -8 \\ -1 & -2 \end{bmatrix}\right)$ has basis $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

The 2-eigenspace $\text{null}\left(\begin{bmatrix} 2 & -8 \\ -1 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 8 & \\ & 2 \end{bmatrix}$.

- Finally, we compute the solution $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$:

$$\begin{aligned} \mathbf{y}(t) &= Pe^{Dt}P^{-1}\mathbf{y}(0) \\ &= \underbrace{\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{8t} & \\ & e^{2t} \end{bmatrix} \begin{pmatrix} -\frac{1}{6} \end{pmatrix} \begin{bmatrix} 1 & -4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\begin{bmatrix} -2e^{8t} & 4e^{2t} \\ e^{8t} & e^{2t} \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 \\ 5 \end{bmatrix}} = \frac{1}{6} \begin{bmatrix} 20e^{2t} - 2e^{8t} \\ 5e^{2t} + e^{8t} \end{bmatrix} \end{aligned}$$

Problem 3.

- (a) Consider the following system of initial value problems:

$$\begin{aligned} y_1'' &= 3y_1' + 2y_2' - 5y_1 & y_1(0) &= 1, \quad y_1'(0) = -2, \quad y_2(0) = 3, \quad y_2'(0) = 0 \\ y_2'' &= y_1' - y_2' + 3y_2 \end{aligned}$$

Write it as a first-order initial value problem in the form $\mathbf{y}' = M\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$.

- (b) Convert the third-order initial value problem

$$y''' = 6y'' - 3y' - 10y, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3$$

to a system of first-order initial value problem.

- (c) Solve the original differential equation in the previous part by solving the system.

Solution.

(a) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 0 & 3 & 2 \\ 0 & 3 & 1 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$$

(b) Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 6y'' - 3y' - 10y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -10y_1 - 3y_2 + 6y_3 \end{cases}$.

In matrix form, this is $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix} \mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(c) First, to compute e^{At} for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -3 & 6 \end{bmatrix}$, we need to diagonalize A .

- The eigenvalues of A are $\lambda = 5, 2, -1$.
- The 5-eigenspace $\text{null}\left(\begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 1 \\ -10 & -3 & 1 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix}$.
- The 2-eigenspace $\text{null}\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -10 & -3 & 4 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.
- The -1 -eigenspace $\text{null}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -3 & 7 \end{bmatrix}\right)$ has basis $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Hence, $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & -1 \end{bmatrix}$.

Then, we compute the solution $\mathbf{y} = e^{At}\mathbf{y}(0)$:

$$\begin{aligned} \mathbf{y} = e^{At}\mathbf{y}(0) &= Pe^{Dt}P^{-1}\mathbf{y}(0) \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \begin{bmatrix} -e^{5t} \\ 20e^{2t} \\ -e^{-t} \end{bmatrix} \\ &= \frac{1}{18} \begin{bmatrix} -e^{5t} + 20e^{2t} - e^{-t} \\ -5e^{5t} + 40e^{2t} + e^{-t} \\ -25e^{5t} + 80e^{2t} - e^{-t} \end{bmatrix} \end{aligned}$$

In particular, the original differential equation is solved by $y(t) = \frac{1}{18}(-e^{5t} + 20e^{2t} - e^{-t})$.

Comment. To compute $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$, we solve $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 2 & -1 \\ 25 & 4 & 1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ to find $\mathbf{v} = \frac{1}{18} \begin{bmatrix} -1 \\ 20 \\ -1 \end{bmatrix}$.

Next comment. Obviously, computations will be more pleasant on the exam.

Problem 4.

- (a) What are the possible Jordan normal forms of a 6×6 matrix with eigenvalues $7, 7, 3, 3, 3, 3$?
- (b) How many different Jordan normal forms are there for a 10×10 matrix with eigenvalues $8, 6, 6, 2, 2, 2, 1, 1, 1, 1$?

Solution.

- (a) There are $2 \cdot 5 = 10$ possibilities:

$$\begin{bmatrix} 7 & & & & & \\ & 7 & & & & \\ & & 3 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & 1 & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & 1 & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix},$$

$$\begin{bmatrix} 7 & 1 & & & & \\ & 7 & & & & \\ & & 3 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & 1 & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}, \begin{bmatrix} 7 & 1 & & & & \\ & 7 & & & & \\ & & 3 & 1 & & \\ & & & 3 & 1 & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}$$

- (b) There are $1 \cdot 2 \cdot 3 \cdot 5 = 30$ possible different Jordan normal forms.

Problem 5. Consider the sequence a_n defined by $a_{n+2} = a_{n+1} + 6a_n$ and $a_0 = 3, a_1 = -1$.

- (a) Determine the next two terms.
- (b) Find an explicit (Binet-like) formula for a_n .
- (c) Determine $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

Solution.

- (a) $a_2 = 17, a_3 = 11$

- (b) The recursion can be translated to $\begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$. We write $A = \begin{bmatrix} 1 & 6 \\ 1 & 0 \end{bmatrix}$.

The characteristic polynomial is: $\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 6 \\ 1 & -\lambda \end{bmatrix}\right) = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$.

Hence, the eigenvalues are $\lambda = 3$ and $\lambda = -2$.

It follows that $a_n = C_1 3^n + C_2 (-2)^n$ and we only need to figure out the values of C_1 and C_2 . Using the two initial conditions, we get two equations:

$$(a_0 =) C_1 + C_2 = 3, (a_1 =) 3C_1 - 2C_2 = -1.$$

Solving, we find $C_1 = 1$ and $C_2 = 2$ so that, in conclusion, $a_n = 3^n + 2 \cdot (-2)^n$.

- (c) It follows from the Binet-like formula that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 3$.

Problem 6.

- (a) Find the best approximation (in the L^2 sense) of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a\sqrt{x}$.

- (b) Find the best approximation (in the L^2 sense) of $f(x) = x$ on the interval $[0, 4]$ using a function of the form $y = a + b\sqrt{x}$.

Solution.

- (a) The best approximation we are looking for is the orthogonal projection of $f(x)$ onto $\text{span}\{\sqrt{x}\}$, where the dot product of functions is

$$\langle f, g \rangle = \int_0^4 f(t)g(t)dt.$$

The orthogonal projection of $f(x)$ onto $\text{span}\{\sqrt{x}\}$ is

$$\frac{\langle f, \sqrt{x} \rangle}{\langle \sqrt{x}, \sqrt{x} \rangle} \sqrt{x}.$$

We compute the two integrals as follows:

$$\begin{aligned} \langle f, \sqrt{x} \rangle &= \int_0^4 f(t)\sqrt{t}dt = \int_0^4 t^{3/2}dt = \left[\frac{2}{5}t^{5/2} \right]_0^4 = \frac{64}{5} \\ \langle \sqrt{x}, \sqrt{x} \rangle &= \int_0^4 \sqrt{t}^2 dt = \int_0^4 t dt = \left[\frac{t^2}{2} \right]_0^4 = 8 \end{aligned}$$

Put together, the best approximation is

$$\frac{\langle f, \sqrt{x} \rangle}{\langle \sqrt{x}, \sqrt{x} \rangle} \sqrt{x} = \frac{64}{5} \cdot \frac{1}{8} \sqrt{x} = \frac{8}{5} \sqrt{x}.$$

- (b) We now are looking for the orthogonal projection of $f(x)$ onto $\text{span}\{1, \sqrt{x}\}$ (with the same dot product as before).

To find an orthogonal basis for $\text{span}\{1, \sqrt{x}\}$, following Gram-Schmidt, we compute

$$\sqrt{x} - \left(\text{projection of } \sqrt{x} \text{ onto } 1 \right) = \sqrt{x} - \frac{\langle \sqrt{x}, 1 \rangle}{\langle 1, 1 \rangle} 1 = \sqrt{x} - \frac{4}{3}.$$

In the last step, we used that

$$\langle 1, 1 \rangle = \int_0^4 1 dt = 4, \quad \langle \sqrt{x}, 1 \rangle = \int_0^4 \sqrt{t} dt = \left[\frac{1}{3/2} t^{3/2} \right]_0^4 = \frac{16}{3}.$$

Hence, $1, \sqrt{x} - \frac{4}{3}$ is an orthogonal basis for $\text{span}\{1, \sqrt{x}\}$.

The orthogonal projection of $f: [0, 4] \rightarrow \mathbb{R}$ onto $\text{span}\{1, \sqrt{x}\} = \text{span}\{1, \sqrt{x} - \frac{4}{3}\}$ therefore is

$$\frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle f, \sqrt{x} - \frac{4}{3} \rangle}{\langle \sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3} \rangle} \left(\sqrt{x} - \frac{4}{3} \right).$$

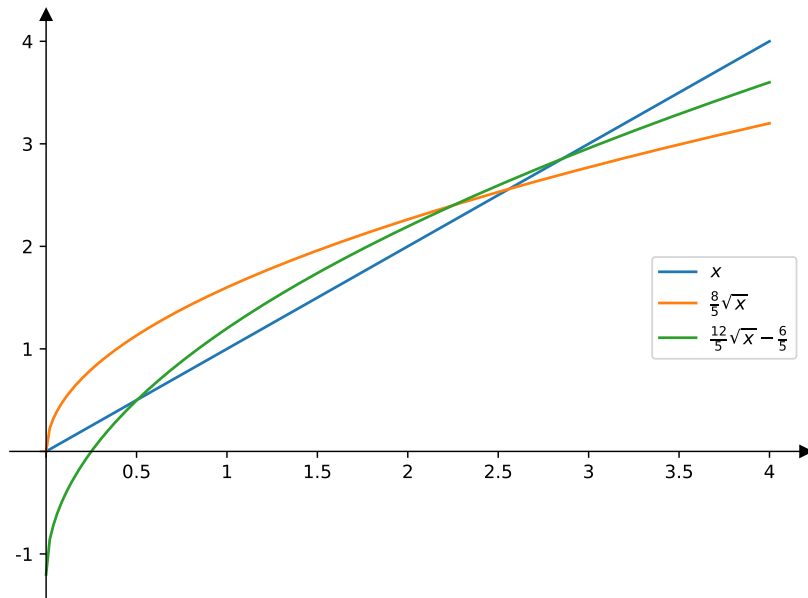
We compute the three new integrals as follows:

$$\begin{aligned} \langle f, 1 \rangle &= \int_0^4 t dt = \left[\frac{t^2}{2} \right]_0^4 = 8 \\ \left\langle f, \sqrt{x} - \frac{4}{3} \right\rangle &= \int_0^4 f(t) \left(\sqrt{t} - \frac{4}{3} \right) dt = \int_0^4 t \left(\sqrt{t} - \frac{4}{3} \right) dt = \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^2 \right]_0^4 = \frac{2}{5} \cdot 32 - \frac{2}{3} \cdot 16 = \frac{32}{5} \\ \left\langle \sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3} \right\rangle &= \int_0^4 \left(\sqrt{t} - \frac{4}{3} \right)^2 dt = \int_0^4 \left(t - \frac{8}{3} \sqrt{t} + \frac{16}{9} \right) dt = \left[\frac{t^2}{2} - \frac{16}{9} t^{3/2} + \frac{16}{9} t \right]_0^4 = 8 - \frac{128}{9} + \frac{64}{9} = \frac{8}{9} \end{aligned}$$

Put together, the best approximation is

$$\frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle f, \sqrt{x} - \frac{4}{3} \rangle}{\langle \sqrt{x} - \frac{4}{3}, \sqrt{x} - \frac{4}{3} \rangle} \left(\sqrt{x} - \frac{4}{3} \right) = \frac{8}{4} + \frac{32}{15} \cdot \frac{9}{8} \left(\sqrt{x} - \frac{4}{3} \right) = \frac{12}{5} \sqrt{x} - \frac{6}{5}.$$

Comment. The plot below confirms the quality of these approximations:



Problem 7.

(a) The eigenvalues of a 5×5 matrix for orthogonally projecting onto a 3-dimensional subspace are .

What are the eigenspaces of that matrix?

(b) Suppose A is the 3×3 matrix of a reflection through a plane (containing the origin).

Then $\det(A) = \text{}$, and the eigenvalues of A are . What are the eigenspaces of A ?

(c) If A has λ -eigenvalue \mathbf{v} , then A^3 has .

(d) If $A = \begin{bmatrix} i & 1+2i \\ 3 & 4 \\ 5i & 6-i \end{bmatrix}$, then its conjugate transpose is $A^* = \text{}$.

(e) The norm of the vector $\mathbf{v} = \begin{bmatrix} 1-i \\ 2i \end{bmatrix}$ is $\|\mathbf{v}\| = \text{}$.

(f) If A is a reflection matrix, then $A^{-1} = \text{}$.

(g) Write down the 2×2 rotation matrix by angle θ .

The 1-eigenspace is $\text{null}(T - 1 \cdot I) = \text{null}\left(\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix}\right)$

To compute a basis, we perform Gaussian elimination (details below):

$$\begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that the 1-eigenspace has basis $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. (Note that its entries add up to $2 + 1 + 1 + 1 = 5$.)

The corresponding equilibrium state is $\frac{1}{5} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$. This is the PageRank vector.

Correspondingly, we rank A the highest, followed by B, C, D which we rank equally.

[In hindsight, can you (at least sort of) see, directly from the diagram, why the PageRank is what it is?]

The full steps of the Gaussian elimination are:

$$\begin{array}{l} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \xrightarrow{\substack{R_2 + \frac{1}{2}R_1 \Rightarrow R_2 \\ R_3 + \frac{1}{2}R_1 \Rightarrow R_3}} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{3}{4} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & -1 \end{bmatrix} \xrightarrow{\substack{R_3 + \frac{1}{3}R_2 \Rightarrow R_3 \\ R_4 + \frac{2}{3}R_2 \Rightarrow R_4}} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{R_4 + R_3 \Rightarrow R_4} \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \\ \begin{array}{l} -1R_1 \Rightarrow R_1 \\ -\frac{4}{3}R_2 \Rightarrow R_2 \\ -\frac{3}{2}R_3 \Rightarrow R_3 \end{array} \xrightarrow{\sim} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_1 + \frac{1}{2}R_3 \Rightarrow R_1 \\ R_2 + \frac{1}{3}R_3 \Rightarrow R_2}} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + \frac{1}{2}R_2 \Rightarrow R_1} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

This was good practice of elimination! However, notice that we can actually find an eigenvector \mathbf{x} with less effort by spelling out the equations: for instance, the second one is just $\frac{1}{2}x_1 - x_2 = 0$. Do that!

Problem 9. Determine an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

- (a) If A is the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- (b) If A is the 3×3 matrix for reflecting through the plane spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. In each case, let W be the plane spanned by the given two vectors. Then A has 1-eigenspace W and -1 -eigenspace W^\perp . We need orthonormal bases for W and W^\perp in order to write down the diagonalization $A = PDP^T$.

Important comment. Note that, if we just use any bases for W and W^\perp , then we would only get a diagonalization of the type $A = PDP^{-1}$.

- (a) Here, $W = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.
- (basis for W) Since the vectors are already orthogonal, we normalize to find that $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an orthonormal basis for W .

- (basis for W^\perp) We can read off that $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ is an orthonormal basis for W^\perp .

Alternatively. If we didn't see that, we can use that $W = \text{col} \left(\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$ so that $W^\perp = \text{null} \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$.

The latter has basis $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$, which becomes $\frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ when normalized.

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & 0 & -1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(b) Here, $W = \text{span} \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

- (basis for W) We apply Gram-Schmidt:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

Normalizing, we find that $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ is an orthonormal basis for W .

- (basis for W^\perp) Since $W = \text{col} \left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)$, we have $W^\perp = \text{null} \left(\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right)$.

Solving the system, $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1/2 \end{bmatrix}$, we find that $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$ is a basis for W^\perp . Normalized: $\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

Alternatively. We could also use $W = \text{col} \left(\begin{bmatrix} 2 & -1 \\ 0 & 5 \\ 1 & 2 \end{bmatrix} \right)$ so that $W^\perp = \text{null} \left(\begin{bmatrix} 2 & 0 & 1 \\ -1 & 5 & 2 \end{bmatrix} \right)$.

In conclusion, we have $A = PDP^T$ with $P = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{30} & -1/\sqrt{6} \\ 0 & 5/\sqrt{30} & -1/\sqrt{6} \\ 1/\sqrt{5} & 2/\sqrt{30} & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.