

Homework Set 10

Problem 1

Example 15. Invert the complex number $5 - 4i$.

Solution.
$$\frac{1}{5 - 4i} = \frac{5 + 4i}{(5 - 4i)(5 + 4i)} = \frac{5 + 4i}{5^2 - (4i)^2} = \frac{5 + 4i}{25 + 16} = \frac{5}{41} + \frac{4}{41}i$$

Problem 2

Example 16. Determine the square of the norm of the vector $\begin{bmatrix} 5 - 5i \\ 1 + 3i \end{bmatrix}$.

Solution.
$$\left\| \begin{bmatrix} 5 - 5i \\ 1 + 3i \end{bmatrix} \right\|^2 = |5 - 5i|^2 + |1 + 3i|^2 = (5^2 + (-5)^2) + (1^2 + 3^2) = 50 + 10 = 60$$

Problem 3

Example 17. Find the best approximation of $f(x) = x^2$ on the interval $[1, 4]$ using a function of the form $y = a\sqrt{x}$.

Solution. Because we are working with functions on $[1, 4]$, the dot product between two functions is

$$\langle f(x), g(x) \rangle = \int_1^4 f(t)g(t)dt.$$

The best approximation is the orthogonal projection of x^2 onto $\text{span}\{\sqrt{x}\}$, which is

$$\begin{aligned} \frac{\langle x^2, \sqrt{x} \rangle}{\langle \sqrt{x}, \sqrt{x} \rangle} \sqrt{x} &= \frac{\int_1^4 t^2 \cdot \sqrt{t} dt}{\int_1^4 \sqrt{t} \cdot \sqrt{t} dt} \cdot \sqrt{x} = \frac{\int_1^4 t^{5/2} dt}{\int_1^4 t dt} \cdot \sqrt{x} \\ &= \frac{\left[\frac{1}{7/2} t^{7/2} \right]_1^4}{\left[\frac{1}{2} t^2 \right]_1^4} \cdot \sqrt{x} = \frac{\frac{2}{7} \cdot 4^{7/2} - \frac{2}{7}}{\frac{1}{2} \cdot 4^2 - \frac{1}{2}} \cdot \sqrt{x} = \frac{\frac{254}{7}}{\frac{15}{2}} \cdot \sqrt{x} = \frac{508}{105} \sqrt{x}. \end{aligned}$$

Problem 4

Example 18. Find the best approximation of $f(x) = x^4$ on the interval $[1, 3]$ using a function of the form $y = a + bx$.

Solution. First, note that this best approximation is the orthogonal projection of x^4 onto $\text{span}\{1, x\}$. However, this orthogonal projection is not simply the projection onto 1 plus the projection onto x . That's because 1 and x are not orthogonal (as functions on $[1, 3]$):

$$\langle 1, x \rangle = \int_1^3 t dt = \left[\frac{1}{2} t^2 \right]_1^3 = \frac{3^2}{2} - \frac{1}{2} = 4 \neq 0.$$

To find an orthogonal basis for $\text{span}\{1, x\}$, following Gram–Schmidt, we compute

$$x - \left(\begin{array}{c} \text{projection of} \\ x \text{ onto } 1 \end{array} \right) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{\int_1^3 t dt}{\int_1^3 1 dt} = x - \frac{4}{2} = x - 2.$$

Hence, $1, x - 2$ is an orthogonal basis for $\text{span}\{1, x\}$.

The desired orthogonal projection of x^4 onto $\text{span}\{1, x\} = \text{span}\{1, x - 2\}$ therefore is

$$\frac{\langle x^4, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle x^4, x - 2 \rangle}{\langle x - 2, x - 2 \rangle} (x - 2) = \frac{\int_1^3 t^4 dt}{\int_1^3 1 dt} + \frac{\int_1^3 t^4 (t - 2) dt}{\int_1^3 (t - 2)^2 dt} (x - 2)$$

We compute the three new integrals:

$$\begin{aligned} \int_1^3 t^4 dt &= \left[\frac{1}{5} t^5 \right]_1^3 = \frac{3^5}{5} - \frac{1}{5} = \frac{242}{5} \\ \int_1^3 (t - 2)^2 dt &= \int_1^3 (t^2 - 4t + 4) dt = \left[\frac{1}{3} t^3 - 2t^2 + 4t \right]_1^3 = 3 - \frac{7}{3} = \frac{2}{3} \\ \int_1^3 t^4 (t - 2) dt &= \int_1^3 (t^5 - 2t^4) dt = \left[\frac{1}{6} t^6 - \frac{2}{5} t^5 \right]_1^3 = \frac{243}{10} - \left(-\frac{7}{30} \right) = \frac{368}{15} \end{aligned}$$

Using these values, the best approximation is

$$\frac{\int_1^3 t^4 dt}{\int_1^3 1 dt} + \frac{\int_1^3 t^4 (t - 2) dt}{\int_1^3 (t - 2)^2 dt} (x - 2) = \frac{242}{5} + \frac{368}{\frac{2}{3}} (x - 2) = \frac{121}{5} + \frac{184}{5} (x - 2) = \frac{184}{5} x - \frac{247}{5}.$$