Example 150. Suppose that a rod of length L is compressed by a force P (with ends being pinned [not clamped] down). We model the shape of the rod by a function $y(x)$ on some interval $[0, L]$. The theory of elasticity predicts that, under certain simplifying assumptions, y should satisfy $EIy'' + Py = 0$, $y(0) = 0$, $y(L) = 0$.

Here, EI is a constant modeling the inflexibility of the rod (E, known as Young's modulus, depends on the material, and I depends on the shape of cross-sections (it is the area moment of inertia)).

In other words, $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$, with $\lambda = \frac{P}{EI}$.

The fact that there is no nonzero solution unless $\lambda = \left(\frac{\pi n}{L}\right)^2$ for some $n = 1, 2, 3, \ldots$, means that buckling can only occur if $P = \left(\frac{\pi n}{L}\right)^2 EI$. In particular, no buckling occurs for forces less than $\frac{\pi^2 EI}{L^2}$. This is known as the critical load (or Euler load) of the rod.

Comment. This is a very simplified model. In particular, it assumes that the deflections are small. (Technically, the buckled rod in our model is longer than L; of course, that's not the case in practice.)

`https://en.wikipedia.org/wiki/Euler%27s_critical_load`

Example 151. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0.$$

Solution. We distinguish three cases:

- **$\lambda < 0$.** The characteristic roots are $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y(3) = A(e^{3r} + e^{-3r})$. Since $e^{3r} + e^{-3r} > 0$, we see that $y(3) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

- **$\lambda = 0$.** The general solution to the DE is $y(x) = A + Bx$. Then $y'(0) = 0$ implies $B = 0$, and it follows from $y(3) = A = 0$ that $\lambda = 0$ is not an eigenvalue.

- **$\lambda > 0$.** The characteristic roots are $\pm i\sqrt{\lambda}$. So, with $r = \sqrt{\lambda}$, the general solution is $y(x) = A\cos(rx) + B\sin(rx)$. $y'(0) = Br = 0$ implies $B = 0$. Then $y(3) = A\cos(3r) = 0$. Note that $\cos(3r) = 0$ is true if and only if $3r = \frac{\pi}{2} + n\pi = \frac{(2n+1)\pi}{2}$ for some integer n. Since $r > 0$, we have $n \geqslant 0$. Correspondingly, $\lambda = r^2 = \left(\frac{(2n+1)\pi}{6}\right)^2$ and $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{(2n+1)\pi}{6}\right)^2$, with $n = 0, 1, 2, \ldots$ with corresponding eigenfunctions $y(x) = \cos\left(\frac{(2n+1)\pi}{6}x\right)$.

Example 152. Suppose $L > 0$. Find all eigenfunctions and eigenvalues of

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0.$$

Solution. To solve this eigenvalue problem, we distinguish three cases:

- $\boldsymbol{\lambda < 0}$. Then, the roots are the real numbers $\pm r = \pm\sqrt{-\lambda}$ and the general solution to the DE is $y(x) = Ae^{rx} + Be^{-rx}$. Then $y'(0) = Ar - Br = 0$ implies $B = A$, so that $y'(L) = A(Le^{Lr} - Le^{-Lr})$. Since $Le^{Lr} - Le^{-Lr} = 0$ only if $r = 0$, we see that $y'(L) = 0$ only if $A = 0$. So there is no solution for $\lambda < 0$.

- $\boldsymbol{\lambda = 0}$. Now, the general solution to the DE is $y(x) = A + Bx$. Then $y'(x) = B$ and we see that $y'(0) = 0$ and $y'(L) = 0$ if and only if $B = 0$. So $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $y(x) = 1$.

- $\boldsymbol{\lambda > 0}$. Now, the roots are $\pm i\sqrt{\lambda}$ and $y(x) = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$. Hence, $y'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$. $y'(0) = B\sqrt{\lambda} = 0$ implies $B = 0$. Then, $y'(L) = -A\sqrt{\lambda}\sin(L\sqrt{\lambda}) = 0$ if and only if $\sin(L\sqrt{\lambda}) = 0$. The latter is true if and only if $L\sqrt{\lambda} = n\pi$ for some integer n. In that case, $\lambda = \left(\frac{n\pi}{L}\right)^2$ and $y(x) = \cos\left(\frac{n\pi}{L}x\right)$.

In summary, we have that the eigenvalues are $\lambda = \left(\frac{\pi n}{L}\right)^2$, $n = 0, 1, 2, 3\ldots$, (why did we include $n = 0$ but excluded $n = -1, -2, \ldots$?!) with corresponding eigenfunctions $y(x) = \cos\left(\frac{\pi n}{L}x\right)$.