

Example 101. Determine $\int \sqrt{1-x^2} dx$.

Solution. We substitute $x = \sin\theta$ (with $\theta \in (-\pi/2, \pi/2)$ so that $\theta = \arcsin(x)$) because then $1-x^2 = \cos^2\theta$. Since $dx = \cos\theta d\theta$, we find

$$\int \sqrt{1-x^2} dx = \int \cos^2\theta d\theta = \dots \text{by parts} \dots = \frac{1}{2}(\cos(\theta)\sin(\theta) + \theta) + C = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin x) + C.$$

See Example 93 for the integration by parts. In the final step, we used $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$ (instead of $\cos(\arcsin(x))$).

Comment. Note that $\int_0^1 \sqrt{1-x^2} dx$ is the area of a quarter of the unit circle (and so has to be $\pi/4$). Using the antiderivative we just computed, we indeed find (since $\sin(\pi/2) = 1$ we have $\arcsin(1) = \pi/2$)

$$\int_0^1 \sqrt{1-x^2} dx = \left[\frac{\arcsin x + x\sqrt{1-x^2}}{2} \right]_0^1 = \frac{\frac{\pi}{2} + 0}{2} - \frac{0+0}{2} = \frac{\pi}{4}.$$

Example 102. Determine $\int \frac{1}{t^2\sqrt{t^2-4}} dt$.

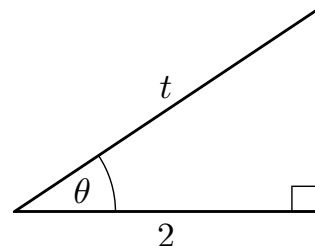
Solution. We substitute $t = 2\sec\theta$ because then $t^2 - 4 = 4(\sec^2\theta - 1) = 4\tan^2\theta$.

Since $\frac{dt}{d\theta} = \frac{d}{d\theta} 2\sec\theta = 2\sec\theta \tan\theta$ (you can work this out from $\sec\theta = \frac{1}{\cos\theta}$), we get

$$\int \frac{1}{t^2\sqrt{t^2-4}} dt = \int \frac{1}{4\sec^2\theta\sqrt{4\tan^2\theta}} 2\sec\theta\tan\theta d\theta = \frac{1}{4} \int \frac{1}{\sec\theta} d\theta = \frac{1}{4} \int \cos\theta d\theta = \frac{1}{4} \sin\theta + C.$$

Our final step consists in simplifying $\sin\theta$ given that $t = 2\sec\theta$.

For this, draw a right-angled triangle with angle θ . To encode the relationship $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{t}{2}$, we assign the hypotenuse length t and the adjacent side length 2 as in the diagram to the right.



By Pythagoras, the opposite side then has length $\sqrt{t^2-4}$. It follows that

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{t^2-4}}{t}.$$

Overall, we have therefore found that

$$\int \frac{1}{t^2\sqrt{t^2-4}} dt = \frac{\sqrt{t^2-4}}{4t} + C.$$

Example 103. Determine $\int \frac{1}{1+x^2} dx$.

Solution. Of course, we already know that $\int \frac{1}{1+x^2} dx = \arctan(x) + C$. On the other hand, in the alternative solution below, we pretend that we didn't.

Solution. We substitute $x = \tan\theta$ because then $1+x^2 = \sec^2\theta$. Since $\frac{dx}{d\theta} = \frac{d}{d\theta} \tan\theta = \sec^2\theta$, we get

$$\int \frac{1}{1+x^2} dx = \int \frac{\sec^2\theta d\theta}{\sec^2\theta} = \int d\theta = \theta + C = \arctan(x) + C.$$