

Trigonometric substitutions

Example 95. Everybody knows that $\cos^2 x + \sin^2 x = 1$.

Divide both sides by $\cos^2 x$ to find $1 + \tan^2 x = \sec^2 x$.

Likewise, dividing by $\sin^2 x$, we find $\cot^2 x + 1 = \csc^2(x)$. However, note that in this identity we cannot have $x = 0$.

if you see	try substituting	because
$a^2 - x^2$ (especially $\sqrt{a^2 - x^2}$)	$x = a \sin \theta$	$a^2 - (a \sin \theta)^2 = a^2 \cos^2 \theta$
$a^2 + x^2$ (especially $\sqrt{a^2 + x^2}$)	$x = a \tan \theta$	$a^2 + (a \tan \theta)^2 = a^2 \sec^2 \theta = \frac{a^2}{\cos^2 \theta}$
and, somewhat less importantly:		
$x^2 - a^2$ (especially $\sqrt{x^2 - a^2}$)	$x = a \sec \theta$	$(a \sec \theta)^2 - a^2 = a^2 \tan^2 \theta$

Note that (by completing the square and doing a simple linear substitution), you can put any quadratic term $ax^2 + bx + c$ into one of these three cases (for instance, $x^2 + 2x + 3 = (x + 1)^2 + 2 = u^2 + 2$ with the simple linear substitution $u = x + 1$).

This is why trigonometric substitution occurs frequently for certain kinds of integrals.

Example 96. Determine $\int \frac{1}{\sqrt{1-x^2}} dx$.

Solution. We substitute $x = \sin \theta$ (with $\theta \in [-\pi/2, \pi/2]$ so that $\theta = \arcsin(x)$) because then $1 - x^2 = \cos^2 \theta$. Since $dx = \cos \theta d\theta$, we find

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} = \int \frac{\cos \theta d\theta}{\sqrt{\cos^2 \theta}} = \int 1 d\theta = \theta + C = \arcsin(x) + C.$$

[Note that in order to conclude $\sqrt{\cos^2 \theta} = \cos \theta$, we used that $\theta \in [-\pi/2, \pi/2]$ and that $\cos \theta \geq 0$ for these values of θ .]

On the other hand. Let's compute the derivative of $\arcsin(x)$ directly from its definition as the inverse function of $\sin(x)$: take the derivative of both sides of $\sin(\arcsin(x)) = x$ to get $\cos(\arcsin(x)) \arcsin'(x) = 1$. Hence

$$\arcsin'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1-x^2}}.$$

Comment. Because the role of $\cos$ and $\sin$ in $\cos^2 \theta + \sin^2 \theta = 1$ is symmetric, we can also substitute $x = \cos \theta$. Do it! When comparing final answers, keep in mind that

$$\arccos(x) = \frac{\pi}{2} - \arcsin(x).$$

This reflects the relationship $\cos(\theta) = \sin(\frac{\pi}{2} - \theta)$.

Excursion: Euler's identity

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y .
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.
Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.
- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.
Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.
- The **real part** of $z = x + iy$ is x and we write $\operatorname{Re}(z) = x$.
 Likewise the **imaginary part** is $\operatorname{Im}(z) = y$.
 Observe that $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 97. (Euler's identity) $e^{ix} = \cos(x) + i \sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy, y(0) = 1$.

[Check that by computing the derivatives and verifying the initial condition! As we did in class.] $\square$

On lots of T-shirts. In particular, with $x = \pi$, we get $e^{i\pi} = -1$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy, y(0) = 1$.

[Check that by computing the derivatives and verifying the initial condition! As we did in class.] $\square$

Comment. It follows that $\cos(x) = \operatorname{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \operatorname{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

In particular, we see from here that $\cos(x) = \cosh(ix)$ and $i \sin(x) = \sinh(ix)$ (or, equivalently, $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i \sin(ix)$).

Example 98. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ (and infinitely many others you have never heard of!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i \sin(2x) \\ e^{ix} e^{ix} &= [\cos(x) + i \sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i \cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i "), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.

Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?

Example 99. Which trig identity hides behind $e^{i(x+y)} = e^{ix}e^{iy}$?

Solution. We observe that

$$\begin{aligned}e^{i(x+y)} &= \cos(x+y) + i \sin(x+y) \\e^{ix}e^{iy} &= [\cos(x) + i \sin(x)][\cos(y) + i \sin(y)] \\&= \cos(x)\cos(y) - \sin(x)\sin(y) + i(\cos(x)\sin(y) + \sin(x)\cos(y)).\end{aligned}$$

Comparing real and imaginary parts, we conclude that

- $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and
- $\sin(x+y) = \cos(x)\sin(y) + \sin(x)\cos(y)$.

Example 100. Which trig identity hides behind $e^{ix}e^{-ix} = 1$?

Solution. Note that

$$\begin{aligned}e^{ix}e^{-ix} &= [\cos(x) + i \sin(x)][\cos(-x) + i \sin(-x)] = [\cos(x) + i \sin(x)][\cos(x) - i \sin(x)] \\&= \cos^2x + \sin^2x.\end{aligned}$$

Hence, $e^{ix}e^{-ix} = 1$ translates into Pythagoras' identity $\cos^2x + \sin^2x = 1$.