

Review. Recall that $dg(x) = g'(x)dx$ (since $\frac{d}{dx}g(x) = g'(x)$). Integration by parts therefore is often written as

$$\int f(x)dg(x) = f(x)g(x) - \int g(x)df(x), \quad \text{or} \quad \int u dv = uv - \int v du.$$

In the latter short form, we have set $u = f(x)$ and $v = g(x)$.

Trigonometric integrals

The following example illustrates that we sometimes have choices when integrating:

Example 89. $\int \sin(x)\cos(x) dx$

Solution. (integration by parts) Integrating by parts with $f(x) = \sin(x)$, $g'(x) = \cos(x)$, $g(x) = \sin(x)$, we get

$$\int \sin(x)\cos(x) dx = \sin^2(x) - \int \cos(x)\sin(x) dx,$$

from which we conclude that $\int \sin(x)\cos(x) dx = \frac{1}{2}\sin^2(x) + C$.

Solution. (substitution) Substitute $u = \sin(x)$, because $du = \cos(x) dx$, to get

$$\int \sin(x)\cos(x) dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(x) + C.$$

Solution. (trig identity) Since $\sin(2x) = 2\cos(x)\sin(x)$, we have

$$\int \sin(x)\cos(x) dx = \frac{1}{2} \int \sin(2x) dx = -\frac{1}{4}\cos(2x) + C.$$

Important comment. Note that $\frac{1}{2}\sin^2(x) \neq -\frac{1}{4}\cos(2x)$ (for instance, plug in $x=0$ to see that). However, the two functions are indeed equal up to a constant (namely, $\frac{1}{2}\sin^2(x) = -\frac{1}{4}\cos(2x) + \frac{1}{4}$) as we can see from the trig identity $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

Example 90. $\int \sin^m(x)\cos^3(x) dx$ (with $m \neq -1, -3$)

Solution. We substitute $u = \sin(x)$, because $du = \cos(x) dx$, to get

$$\begin{aligned} \int \sin^m(x)\cos^3(x) dx &= \int u^m \cos^2(x) du = \int u^m (1 - \sin^2(x)) du = \int u^m (1 - u^2) du \\ &= \frac{u^{m+1}}{m+1} - \frac{u^{m+3}}{m+3} + C = \frac{\sin^{m+1}(x)}{m+1} - \frac{\sin^{m+3}(x)}{m+3} + C. \end{aligned}$$

One exponent may also be negative (in the next example, we integrate $[\sin(x)]^1 [\cos(x)]^{-1}$).

Example 91. Determine $\int \tan(x) \, dx$.

Solution. $\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx$, so we substitute $u = \cos(x)$ (then $du = -\sin(x) \, dx$) to get

$$\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx = - \int \frac{du}{u} = -\ln|u| + C = -\ln|\cos(x)| + C = \ln|\sec(x)| + C.$$

Solution. (harder—only for practice) For some exercise in substituting, we can also substitute $u = \sin(x)$ (but can you explain how we can tell beforehand that $u = \cos(x)$ should be the better choice?). Then $du = \cos(x) \, dx$ or, equivalently, $dx = \frac{1}{\cos(x)} du$, so that we get

$$\int \tan(x) \, dx = \int \frac{\sin(x)}{\cos(x)} \, dx = \int \frac{u}{\cos^2(x)} du = \int \frac{u}{1 - \sin^2(x)} du = \int \frac{u}{1 - u^2} du.$$

We now substitute $v = 1 - u^2$ (so that $dv = -2u \, du$) to get

$$\begin{aligned} \int \tan(x) \, dx &= \int \frac{u}{1 - u^2} du = -\frac{1}{2} \int \frac{dv}{v} = -\frac{1}{2} \ln|v| + C = -\frac{1}{2} \ln|1 - u^2| + C \\ &= -\frac{1}{2} \ln|1 - \sin^2(x)| + C = -\frac{1}{2} \ln|\cos^2(x)| + C = -\ln|\cos(x)| + C \end{aligned}$$

as earlier.

The strategy in the previous problem works whenever we have an odd power of cosine:

Example 92. Describe how we can determine $\int \sin^m(x) \cos^{2k+1}(x) \, dx$.

Solution. Again, we substitute $u = \sin(x)$, because $du = \cos(x) \, dx$, to get

$$\int \sin^m(x) \cos^{2k+1}(x) \, dx = \int u^m \cos^{2k}(x) \, du = \int u^m (1 - \sin^2(x))^k \, du = \int u^m (1 - u^2)^k \, du.$$

For a given integer $k \geq 0$, we can now multiply out the term $(1 - u^2)^k$. Each resulting term (after multiplying with u^m) can then be integrated using the power rule (as in the previous example).

Extrapolating this strategy, we can integrate the following products of trigonometric function:

- $\int \sin^m(x) \cos^n(x) \, dx$, with $n = 2k + 1$ odd, can be evaluated by substituting $u = \sin(x)$.
See previous example!
- $\int \sin^m(x) \cos^n(x) \, dx$, with m odd, can be likewise evaluated by substituting $u = \cos(x)$.
- $\int \sin^m(x) \cos^n(x) \, dx$, with both m, n even, can be reduced via

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}, \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

[Then, multiply out the integrand. The resulting integrals have smaller exponents, and we (recursively) apply our strategy to each of them (if the $2x$ bothers you, substitute $u = 2x$).]

Example 93. Determine $\int \cos^2(x) dx$.

Solution. (trig identity) Since both exponents are even (the exponent of $\sin(x)$ is 0, which is even), we use the trig identity $\cos^2(x) = \frac{1 + \cos(2x)}{2}$:

$$\int \cos^2(x) dx = \int \frac{1 + \cos(2x)}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C.$$

Solution. (integration by parts—only for practice) We choose $f(x) = \cos(x)$ and $g'(x) = \cos(x)$ (so that $g(x) = \sin(x)$) to get

$$\int \cos^2(x) dx = \cos(x)\sin(x) + \int \sin^2(x) dx = \cos(x)\sin(x) + \int (1 - \cos^2(x)) dx.$$

(In the final step, we used that $\cos^2(x) + \sin^2(x) = 1$ to rewrite the $\sin^2(x)$ in terms of $\cos^2(x)$.) Note that our integral appears on both sides. Solving for it, we conclude that

$$\int \cos^2(x) dx = \frac{1}{2}(\cos(x)\sin(x) + x) + C.$$

Our final answer looks different at first glance but is the same because $\sin(2x) = 2\cos(x)\sin(x)$.

Example 94. Determine $\int \cos^2(x)\sin^2(x) dx$.

Solution. Since both exponents are even, we use the trig identities $\cos^2(x) = \frac{1 + \cos(2x)}{2}$, $\sin^2(x) = \frac{1 - \cos(2x)}{2}$:

$$\int \cos^2(x)\sin^2(x) dx = \int \frac{1 + \cos(2x)}{2} \cdot \frac{1 - \cos(2x)}{2} dx = \frac{1}{4} \int (1 - \cos^2(2x)) dx = \frac{1}{4}x - \frac{1}{4} \int \cos^2(2x) dx.$$

We now use $\cos^2(x) = \frac{1 + \cos(2x)}{2}$ again (or we could use Example 93) to find

$$\int \cos^2(2x) dx = \int \frac{1 + \cos(4x)}{2} dx = \frac{1}{2}x + \frac{1}{8}\sin(4x) + B.$$

Combined, we have (we rename the constant of integration to absorb the factor of $-1/4$)

$$\int \cos^2(x)\sin^2(x) dx = \frac{1}{4}x - \frac{1}{4} \left(\frac{1}{2}x + \frac{1}{8}\sin(4x) \right) + C = \frac{1}{8}x - \frac{1}{32}\sin(4x) + C.$$