

Review. By integrating the product rule, we get the formula for integration by parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Example 85. (review) Determine $\int x e^x dx$.

- To evaluate the integral, we should choose $f(x) = x$ and $g'(x) = e^x$, because $f'(x)$ becomes easier while $g(x)$ stays the same. Do it! We find

$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = x e^x - e^x + C.$$

- If, on the other hand, we decide to choose $f(x) = e^x$ and $g'(x) = x$, then we obtain

$$\int x e^x dx = \frac{1}{2} x^2 e^x - \int \frac{1}{2} x^2 e^x dx.$$

While certainly correct, we actually ended up with a more difficult integral.

[On the other hand, because we know $\int x e^x dx$, this means we now also know $\int x^2 e^x dx$.]

Example 86. Determine $\int_0^1 x^2 e^{3x} dx$.

Solution. We do integration by parts with $f(x) = x^2$ and $g'(x) = e^{3x}$ so that $f'(x) = 2x$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x e^{3x} dx.$$

We now do integration by parts again with $f(x) = x$ and $g'(x) = e^{3x}$ so that $f'(x) = 1$ and $g(x) = \frac{1}{3}e^{3x}$ to get

$$\int x e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx = \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}.$$

Taken together, this means

$$\int x^2 e^{3x} dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left(\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} \right) = \left(\frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right) e^{3x}.$$

In particular,

$$\int_0^1 x^2 e^{3x} dx = \left[\left(\frac{1}{3} x^2 - \frac{2}{9} x + \frac{2}{27} \right) e^{3x} \right]_0^1 = \left(\frac{1}{3} - \frac{2}{9} + \frac{2}{27} \right) e^3 - \frac{2}{27} = \frac{5}{27} e^3 - \frac{2}{27}.$$

Alternatively. While doing integration by parts, we can carry the bounds along. For instance, for the first step,

$$\int_0^1 x^2 e^{3x} dx = \left[\frac{1}{3} x^2 e^{3x} \right]_0^1 - \frac{2}{3} \int_0^1 x e^{3x} dx = \frac{1}{3} e^3 - \frac{2}{3} \int_0^1 x e^{3x} dx.$$

For practice, do the second step likewise to get the same final answer as before!

Example 87. Determine $\int e^x \cos(x) \, dx$.

Solution. We will need to integrate by parts twice. First, let $f(x) = \cos(x)$ and $g'(x) = e^x$ so that $f'(x) = -\sin(x)$ and $g(x) = e^x$ (the other way around works as well—see below!) to get

$$\int e^x \cos(x) \, dx = e^x \cos(x) + \int e^x \sin(x) \, dx.$$

The new integral is of the same level of difficulty, so it might seem like we haven't gained anything. But don't give up yet! Instead, integrate by parts again with $f(x) = \sin(x)$ and $g'(x) = e^x$ to arrive at

$$\int e^x \cos(x) \, dx = e^x \cos(x) + \int e^x \sin(x) \, dx = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) \, dx.$$

We can now solve for $\int e^x \cos(x) \, dx$ and find $\int e^x \cos(x) \, dx = \frac{1}{2}(e^x \sin(x) + e^x \cos(x))$.

Solution. (variation) We proceed as before but now let $f(x) = e^x$ and $g'(x) = \cos(x)$ to get

$$\int e^x \cos(x) \, dx = e^x \sin(x) - \int e^x \sin(x) \, dx.$$

Again, we once more integrate by parts: choosing $f(x) = e^x$ and $g'(x) = \sin(x)$, we arrive at

$$\int e^x \cos(x) \, dx = e^x \sin(x) - \int e^x \sin(x) \, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) \, dx.$$

As before, we can then solve for $\int e^x \cos(x) \, dx$ to find $\int e^x \cos(x) \, dx = \frac{1}{2}(e^x \sin(x) + e^x \cos(x))$.

We next discuss integrals of products of trig functions. The following is an example that we are already familiar with:

Example 88. (review/preview) $\int \sin^m(x) \cos(x) \, dx$ (with $m \neq -1$)

Solution. We substitute $u = \sin(x)$, because then $du = \cos(x) \, dx$, to get

$$\int \sin^m(x) \cos(x) \, dx = \int u^m \, du = \frac{1}{m+1} u^{m+1} + C = \frac{\sin^{m+1}(x)}{m+1} + C.$$