

Review. (from midterm exam) Here are two ways to determine $\int \frac{dx}{2x}$. Which is correct?

(a) $\int \frac{dx}{2x} = \frac{1}{2} \int \frac{dx}{x} = \frac{1}{2} \ln|x| + C$

(b) We substitute $u = 2x$ (so that $du = 2dx$) to get:

$$\int \frac{dx}{2x} = \int \frac{\frac{1}{2} du}{u} = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|2x| + C$$

Solution. Both are correct! The answers look different but they only differ by a constant because

$$\frac{1}{2} \ln|2x| = \frac{1}{2} \ln(2|x|) = \frac{1}{2} (\ln(2) + \ln|x|).$$

Integration by parts

If we integrate both sides of the product rule, we obtain the following:

$$\begin{aligned} (fg)' &= f'g + fg' \\ \text{antiderivative} \rightsquigarrow f(x)g(x) &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \end{aligned}$$

If we then solve for one of the two integrals, we get:

(integration by parts)

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

The following shorthand is very common as well:

$$\int u dv = uv - \int v du$$

Here, $u = f(x)$, $v = g(x)$ so that $du = f'(x)dx$ and $dv = g'(x)dx$.

Example 81. Determine $\int x \cos(x) dx$.

Solution. We choose $f(x) = x$ and $g'(x) = \cos(x)$, so that $g(x) = \sin(x)$ (note that we are free to choose the simplest antiderivative for $g(x)$), to get

$$\int x \cos(x) dx = x \sin(x) - \int 1 \cdot \sin(x) dx = x \sin(x) + \cos(x) + C.$$

Example 82. Determine $\int x e^{2x} dx$.

Solution. We choose $f(x) = x$ and $g'(x) = e^{2x}$, so that $g(x) = \frac{1}{2}e^{2x}$, to get

$$\int x e^{2x} dx = x \cdot \frac{1}{2} e^{2x} - \int 1 \cdot \frac{1}{2} e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C = \left(\frac{1}{2} x - \frac{1}{4} \right) e^{2x} + C.$$

Example 83. Determine $\int \ln(x) \, dx$.

Solution. We choose $f(x) = \ln(x)$ and $g'(x) = 1$, so that $g(x) = x$, to get

$$\int \ln(x) \cdot 1 \, dx = \ln(x) \cdot x - \int \frac{1}{x} \cdot x \, dx = x \ln(x) - x + C.$$

Example 84. Substitute $u = \ln(x)$ in the previous integral. What do you get?

Solution. If $u = \ln(x)$ then $du = \frac{1}{x} dx$ so that $dx = x du = e^u du$ (in the last step, we used that $x = e^u$).

We therefore get $\int \ln(x) \, dx = \int u e^u \, du$.

By Example 82 we know $\int u e^u \, du = (u - 1)e^u + C$ so that

$$\int \ln(x) \, dx = \int u e^u \, du = (u - 1)e^u + C = (\ln(x) - 1)e^{\ln(x)} + C = (\ln(x) - 1)x + C,$$

which matches what we obtained in Example 83.

Comment. We can also start by writing $x = e^u$ so that we immediately get $dx = e^u du$. It depends on the integral, which of the two approaches is algebraically simpler.